可積分系および
モノドロミー保存系の
量子化と離散化について

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1. KP → Soliton → Isomonodromic

∞-dim. system:

(多成分) KP $\quad (GL_\infty^n + \text{Virasoro})$

↓ reduction

Soliton hierarchy $\quad (\hat{G} + \text{Virasoro})$

- KdV, Boussinesq, DS, …
- NLS, AKNS, ZS, …
- SG, affine Toda field, …

↓ reduction (string eq.)

finite dim.:

Isomonodromic deformation

ZS → JMU

mDS → NY

KP	Heisenberg flow	Virasoro sym.	Bäcklund 変換
	↓	↓	↓
Soliton	Heisenberg flow	Virasoro sym.	Bäcklund 変換
	↓	↓	↓
Isomonod.	不確定特異点における形式解のdataの変形	特異点の位置(および R.面)の変形	affine Weyl 群作用, Schlesinger 変換, ……

1.1. modified Drinfeld-Sokolov hier.

$\mathfrak{g} = $ (loop alg. of $\mathfrak{gl}_m(\mathbb{C})$), principal gradation

$\mathfrak{g}_+ := \mathfrak{g}_{\geq 0}$, $\mathfrak{g}_- := \mathfrak{g}_{<0}$, $G_\pm := \exp \mathfrak{g}_\pm$

$$\Lambda = \Lambda(z) := \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ z & & & 0 \end{bmatrix}, \quad \Lambda_i := \Lambda^i \quad \begin{pmatrix} \text{principal} \\ \text{Heisenberg} \end{pmatrix}$$

$x = x_-^{-1} x_+ \quad (x_\pm \in G_\pm)$

$$\frac{\partial x}{\partial t_i} = \Lambda_i x, \; i > 0 \quad \text{(Heisenberg flow)}$$

$$\Updownarrow$$

$\Psi := x_- e^{\sum t_i \Lambda_i}$, $\Phi := x_+ z^A$, $A = \begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_m \end{bmatrix}$

$B_i := (x_- \Lambda_i x_-^{-1})_+ \in \mathfrak{g}_+$ とおくと

$$\frac{\partial \Psi}{\partial t_i} = B_i \Psi, \quad \frac{\partial \Phi}{\partial t_i} = B_i \Phi, \quad i > 0.$$

$$\Updownarrow$$

$$\left[\frac{\partial}{\partial t_i} - B_i, \frac{\partial}{\partial t_j} - B_j\right] = 0, \quad i, j > 0$$

(modified DS hierarchy)

1.2. mDS の Weyl 群対称性

$W \subset G$ ($W(A^{(1)}_{m-1})$ の G へのもちあげ), $g \in W$

$$\frac{\partial x}{\partial t_i} = \Lambda_i x \xrightarrow{x = x_-^{-1} x_+} \text{mDS hierarchy}$$

$$x \mapsto \tilde{x} = x g^{-1} \xrightarrow{\tilde{x} = \tilde{x}_-^{-1} \tilde{x}_+} \text{Bäcklund 変換}$$

$b_g := (x_+ g^{-1} x_+^{-1})_- \in G_-$ とおくと,

$$\Psi \mapsto \tilde{\Psi} := \tilde{x}_- e^{\sum t_i \Lambda_i} = b_g \Psi,$$

$$\Phi \mapsto \tilde{\Phi} := \tilde{x}_+ g z^A = b_g x_+ z^{\tilde{A}},$$

$$z^A \longmapsto z^{\tilde{A}} = g z^A g^{-1}$$

$$\frac{\partial}{\partial t_i} - B_i \mapsto \frac{\partial}{\partial t_i} - \tilde{B}_i = b_g \circ \left(\frac{\partial}{\partial t_i} - B_i \right) \circ b_g^{-1}$$

$$A = \begin{bmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_m \end{bmatrix} \mapsto \tilde{A} = \begin{bmatrix} \tilde{\varepsilon}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{\varepsilon}_m \end{bmatrix}$$

は ε_i たちの置換と整数シフト.

1.3. mDS の similarity reduction

Similarity reduction の条件:

(1) $t_r \neq 0, \quad 0 = t_{r+1} = t_{r+2} = \cdots$

(2) $H_p := \frac{1}{2}\mathrm{diag}(m-1, m-3, \ldots, -m+3, -m+1)$,

$$\left(mz\frac{\partial}{\partial z} + H_p\right)\Psi = M^{(\infty)}\Psi, \quad M^{(\infty)} \in \mathfrak{g},$$

$$\left(mz\frac{\partial}{\partial z} + H_p\right)\Phi = M^{(0)}\Phi, \quad M^{(0)} \in \mathfrak{g}$$

とすると,

$$M^{(\infty)} = M^{(0)}.$$

以下, $M := M^{(\infty)} = M^{(0)}$.

(*) $\left(mz\frac{\partial}{\partial z} + H_p\right)Y = MY.$

mDS	$t_1, \ldots, t_r$ の変化	Weyl群作用
	$\downarrow$	$\downarrow$
(*)	Isomonodromic deformation	Bäcklund 変換 (パラメーター ε_i たちの置換とシフト)

1.4. $r=2 \longrightarrow$ Noumi-Yamada system

$$M = \begin{bmatrix} \varepsilon_1 & f_1 & 1 & & \\ & \varepsilon_2 & f_2 & \ddots & \\ & & \varepsilon_3 & \ddots & 1 \\ z & & & \ddots & f_{m-1} \\ zf_m & z & & & \varepsilon_m \end{bmatrix}$$

← r=2の場合のMの形

$$B_1 = \begin{bmatrix} q_1 & 1 & & \\ & q_2 & \ddots & \\ & & \ddots & 1 \\ z & & & q_m \end{bmatrix}$$

(以下, カンタンのため $m=2g+1$ (奇数))

$$\left[\frac{\partial}{\partial t_1} - B_1,\ mz\frac{\partial}{\partial z} + H - M\right] = 0 \Leftrightarrow \text{NY system}$$

$m=3 \Longleftrightarrow$ Painlevé IV

NY system については

Hamiltonian str. がわかっている.

(Noumi-Yamada, math.QA/9808003)

- NY system の量子化

  (名古屋創, 数理研講究録 (2003) に出る予定)

- 一般化 DS の similarity reduction

  (Kikuchi-Ikeda-Kakei, J. Phys. A36 (2003)

  菊地哲也・筧三郎, 数理研講究録 (2003))

同じ

2. Quantization of isomonodromic systems

- $L = \frac{\partial}{\partial z} - \sum_{k=1}^{n} \frac{A_k}{z - z_k} \xmapsto{\text{quantize}}$ KZ equation

  (Schlesinger eq.) (Reshetikhin (1992), Harnad (1994))

- $L = \frac{\partial}{\partial z} - \left[\mathrm{diag}(t_1, \dots, t_m) + \sum_{k=1}^{n} \frac{A_k}{z - z_k} \right]$

  $\longmapsto$ generalized KZ eq.

  (Babujian-Kitaev (1998))

- $L = mz\frac{\partial}{\partial z} + H - M, \quad M = \begin{bmatrix} \varepsilon_1 & f_1 & 1 & & & \\ & \varepsilon_2 & f_2 & \ddots & & \\ & & \varepsilon_3 & \ddots & \ddots & 1 \\ z & & & \ddots & \ddots & f_{m-1} \\ zf_m & z & & & & \varepsilon_m \end{bmatrix}$

  (NY system)

  $\longmapsto$ <u>名古屋創</u>による量子化

- Kajiwara-Noumi-Yamada (nlin.SI/0012063)の

  q差分版のNY systemへのWeyl群作用

  $\longmapsto$ <u>長谷川浩司</u>による量子化

2.1. NY systemの量子化

カンタンのため $m=2g+1$ (奇数)とする.

Commutation relations:

$$[f_i, f_{i\pm1}] = \mp\hbar, \quad f_{i+m} = f_i,$$

$$[\varepsilon_i, \varepsilon_j] = [\varepsilon_i, f_j] = 0, \quad \varepsilon_{i+m} = \varepsilon_i - \underset{\underset{\mathbb{C}}{\cap}}{\varepsilon}.$$

$$\alpha_i := \varepsilon_i - \varepsilon_{i+1}, \quad \alpha_{i+m} = \alpha_i$$

$m=3$のときのHamiltonian:

$$h_1 = f_1 f_2 f_3 \underbrace{- \hbar f_2} + \varepsilon_3 f_1 + \varepsilon_1 f_2 + \varepsilon_2 f_3$$

この項は消せない

しかし, $m \geq 5$では消せる(名古屋)

方程式:

$$\dot{f}_i = f_i f_{i+2} - f_{i+1} f_i + \alpha_i \quad \leftarrow m=3$$

$$= \hbar^{-1}[h_1, f_i] + \delta_{i,0}\,\varepsilon.$$

Weyl群作用:

$$s_i(\varepsilon_i) = \varepsilon_{i+1}, \quad s_i(\varepsilon_{i+1}) = \varepsilon_i,$$

$$s_i(f_{i\pm1}) = f_{i\pm1} \mp \frac{\alpha_i}{f_i}.$$

(以上は名古屋の仕事)

2.2. Weyl群作用の $x \mapsto \tilde{S}_i x \tilde{S}_i^{-1}$ の形での実現

Relations：

$$r_i^2 = 1,\quad r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1},\ \dots,$$

$$r_i \varepsilon_i r_i^{-1} = \varepsilon_{i+1},\quad r_i \varepsilon_{i+1} r_i^{-1} = \varepsilon_i,\ \dots,$$

$$r_i F_j r_i^{-1} = F_j.$$

$\tilde{S}_i$ の def.：

$$\tilde{S}_i := f_i^{\alpha_i/\hbar} r_i.$$

このとき，

$$\tilde{S}_i^2 = 1,\quad \tilde{S}_i \tilde{S}_{i+1} \tilde{S}_i = \tilde{S}_{i+1} \tilde{S}_i \tilde{S}_{i+1}.$$

Weyl群作用は次のように書ける：

$$s_i(x) = \tilde{S}_i x \tilde{S}_i^{-1} \quad (x = \varepsilon_i, f_i).$$

2.3. 量子 q 差分版の Weyl 群作用

Relations:

$$F_{i+1}F_i = qF_iF_{i+1},\quad F_{i+m}=F_i,$$

$$a_iF_j=F_ja_i,\quad a_ia_j=a_ja_i,\quad a_{i+m}=a_i,$$

$$r_ia_ir_i^{-1}=a_i^{-1},\quad r_ia_{i\pm1}r_i^{-1}=a_ia_{i\pm1},$$

$$r_iF_jr_i^{-1}=F_j.$$

S_i の def.

$$S_i:=\Psi_ir_i,\quad \Psi_i=\frac{(qF_i;q)_\infty(F_i^{-1};q)_\infty}{(qa_iF_i;q)_\infty(a_iF_i^{-1};q)_\infty},$$

$$S_iS_{i+1}S_i=S_{i+1}S_iS_{i+1}.\qquad (x;q)_\infty=(1-x)(1-qx)(1-q^2x)\cdots$$

Weyl 群作用:

$$s_i(x):=S_i\overset{x}{*}S_i^{-1}\quad (x=a_i,F_i).$$

$$s_i(a_i)=a_i^{-1},\quad s_i(a_{i\pm1})=a_ia_{i\pm1},$$

$$s_i(F_{i-1})=\frac{1-a_iF_i}{a_i-F_i}F_{i-1},\quad s_i(F_{i+1})=F_{i+1}\frac{a_i-F_i}{1-a_iF_i},$$

以上は長谷川の仕事. A型以外もできている.

3. Discretization and lattice systems

3.1. KZ world (ultralocal case)

微分

$$z\frac{\partial}{\partial z}Y = \left[\sum_{k=1}^{n} L_k(z)\right] Y, \quad L_k(z) = \frac{A_k}{z - z_k}$$

Schlesinger eq. $\longmapsto$ KZ eq. (affine Lie alg.)

差分

$$Y(z+\varepsilon) = L_1(z) \cdots L_n(z) Y(z),$$

$$L_k(z) = z + A_k, \quad L_{k+n}(z) = L_k(z-\varepsilon)$$

quasi n-periodicity

差分 Schlesinger eq. (Borodin (2002)) $\longmapsto$ 差分 KZ eq. (Yangian)

q差分

$$Y(qz) = L_1(z) \cdots L_n(z) Y(z)$$

q差分 Schlesinger eq. $\longmapsto$ q差分 KZ eq. (U_q)

問題 Schlesinger 変換の量子化?

3.2. dressing chain の量子化

古典の場合については、

Shabat - Yamilov (1991),
Veselov - Shabat (1993),
V. E. Adler (1994),
Takasaki (2002), ...

$m = 2g+1$, <u>quasi m-periodicity</u> を仮定

$$
\begin{tikzcd}
\text{classical NY system} \arrow[r, leftrightarrow, "\text{同値}"] \arrow[d, "\text{量子化}"'] & \text{classical dressing chain} \arrow[d, "\text{量子化}"] \\
\text{quantum NY system (名古屋)} \arrow[r, leftrightarrow, "\text{同値}"'] & \text{quantum dressing chain}
\end{tikzcd}
$$

$$f_i = v_i + v_{i+1} \longleftrightarrow v_i = \frac{1}{2}\left(f_i - f_{i+1} + f_{i+2} - \cdots + f_m\right)$$

$$[f_i, f_{i\pm 1}] = \mp \hbar \qquad\qquad [v_i, v_j] = (-1)^{j-i}\hbar \quad (i<j<i+m)$$

local comm. rel. $\qquad\qquad$ non-local!

quantum dressing chain

$$[v_i, v_j] = (-1)^{j-i}\hbar \quad (i<j<i+m), \quad v_{i+m} = v_i.$$

$$\varepsilon_{i+m} = \varepsilon_i - \varepsilon, \quad \alpha_i := \varepsilon_i - \varepsilon_{i+1},$$

方程式： $\dot{v}_i + \dot{v}_{i+1} = v_i^2 - v_{i+1}^2 + \alpha_i \quad \longleftarrow \; (*)$

Lax表示：

$$L_i = L_i(z) := \begin{bmatrix} v_i & 1 \\ z+\varepsilon_i+v_i^2 & v_i \end{bmatrix} \quad \leftarrow \text{local L-op}$$

$$U_i = U_i(z) := \begin{bmatrix} 0 & 1 \\ z+\varepsilon_i+u_i & 0 \end{bmatrix} \quad (u_i \text{は後で消す})$$

$$\dot{L}_i = U_i L_i - L_i U_{i+1} \iff (*)$$

$f_i := v_i + v_{i+1}$ とおくと, $m=3$ のとき,

$$\dot{f}_i = f_i f_{i+2} - f_{i+1} f_i + \alpha_i \iff (*)$$

$\mathrm{tr}(L_1 \cdots L_m)$ $(m=2g+1)$ は z について g 次式.

Hamiltonian

$$h_1 = (\mathrm{tr}(L_1 \cdots L_m) \text{の } z^{g-1} \text{の係数})$$

(これは名古屋の Hamiltonian と一致)

$m=3$ のとき,

$$h_1 = f_1 f_2 f_3 - \hbar f_2 + \varepsilon_3 f_1 + \varepsilon_1 f_2 + \varepsilon_2 f_3$$

Weyl群作用

$S_i := f_i^{\alpha_i/\hbar} r_i$ とおき,

$$s_i(x) := S_i x S_i^{-1} \quad (x = \varepsilon_i, v_i)$$

と定めると,

$$s_i(\varepsilon_i) = \varepsilon_{i+1},\ s_i(\varepsilon_{i+1}) = \varepsilon_i,$$

$$s_i(v_i) = v_i + \frac{\alpha_i}{f_i},\ s_i(v_{i+1}) = v_{i+1} - \frac{\alpha_i}{f_i}.$$

$\tilde{x} := s_i(x)$, $\tilde{L}_j := \begin{bmatrix} \tilde{v}_j & 1 \\ z + \tilde{\varepsilon}_j + \tilde{v}_j^2 & \tilde{v}_j \end{bmatrix}$ とおくと,

(1) $\tilde{L}_j = L_j \quad (j \neq i, i+1)$,

(2) $\tilde{L}_i \tilde{L}_{i+1} = L_i L_{i+1}$,

$\det \tilde{L}_i = \det L_{i+1}$,

$\det \tilde{L}_{i+1} = \det L_i$.

L_iの世界では commutation relation は non-local だが,
Weyl群作用は local な形をしている.

3.3. 量子q差分版のWeyl群作用と lattice system

長谷川の量子q差分版Weyl群作用を lattice system で実現する $(m=2g+1)$

$$x_i x_j = q^{(-1)^{j-i}} x_j x_i \quad (i<j<i+m)$$

$$e_i e_j = e_j e_i, \quad e_i x_j = x_j e_i.$$

local L-operator:

$$L_i = \begin{bmatrix} x_i & 1 \\ z & e_i x_i^{-1} \end{bmatrix}$$

Weyl群作用:

$$s_i(e_i) = e_{i+1}, \quad s_i(e_{i+1}) = e_i,$$

$$s_i(x_i) = \frac{x_i x_{i+1} + e_{i+1}}{x_i x_{i+1} + e_i} x_i,$$

$$s_i(x_{i+1}) = x_{i+1} \frac{x_i x_{i+1} + e_i}{x_i x_{i+1} + e_{i+1}}.$$

$\tilde{X} := s_i(X)$ とおくと、

(1) $\tilde{L}_j = L_j \quad (j \neq i, i+1)$

(2) $\tilde{L}_i \tilde{L}_{i+1} = L_i L_{i+1}, \quad \det \tilde{L}_i = \det L_{i+1}$
$\det \tilde{L}_{i+1} = \det L_i$

長谷川の F_i との関係:

$$F_i = -\frac{x_i x_{i+1}}{\sqrt{e_i e_{i+1}}}, \quad a_i = \sqrt{\frac{e_i}{e_{i+1}}}.$$

4. 様々な2×2の表

	モノドロミー保存系	可積分系
量子	KZ eq.	量子 Gaudin model
古典	Schlesinger eq.	古典 Gaudin model

	(q) 差分	微分
量子	(q) 差分 KZ eq.	KZ eq.
古典	(q) 差分 Schlesinger eq.	Schlesinger eq.

	q 差分	微分
モノドロミー保存系	q 差分 KZ eq.	KZ eq.
可積分系	XXZ Heisenberg	XXZ Gaudin

上に登場する系はどれも ultralocal.

	微分	差分
ソリトン系	modified Drinfeld-Sokolov	infinite dressing chain
モノドロミー保存系	Noumi-Yamada system	quasi-periodic dressing chain

	差分	q差分
lattice system	$L_i = \begin{bmatrix} v_i & 1 \\ z+\varepsilon_i+v_i^2 & v_i \end{bmatrix}$	$L_i = \begin{bmatrix} x_i & 1 \\ z & e_i x_i^{-1} \end{bmatrix}$
Weyl群作用	quasi-periodic dressing chainの量子化へのWeyl群作用	長谷川の量子q差分版のWeyl群作用と一致

他にもいくらでもこのような表を書くことができる。