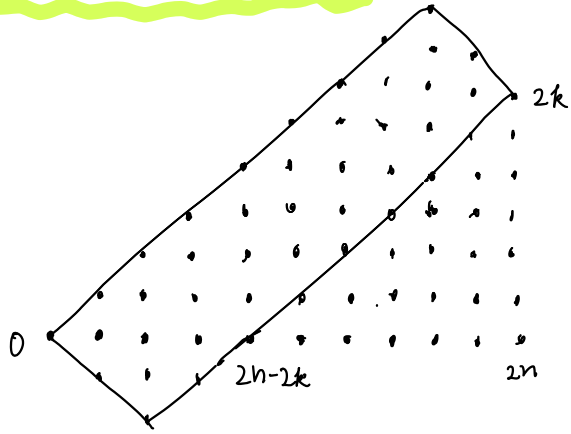
個数大文字, 確率小文字

$n=5, k=3$

まとめ



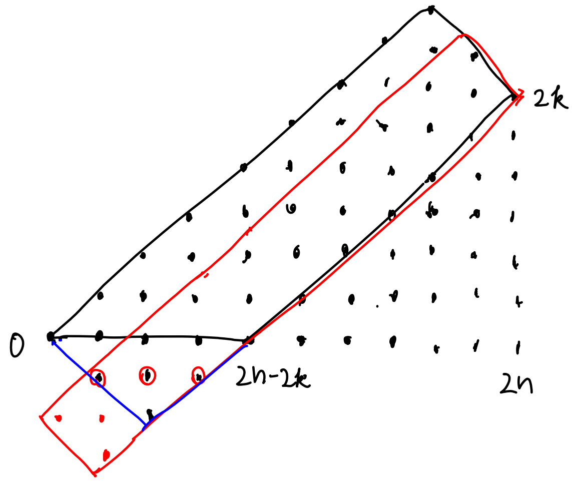
$\mapsto U_{2n,2k} = \binom{2n}{n-k}$, $U_{2n} := U_{2n,0}$

$\mapsto u_{2n,2k} = \frac{1}{2^{2n}} U_{2n,2k}$

$u_{2n} := u_{2n,0}$

$f'_{2n,2k} = u_{2n,2k} - u_{2n,2k+2}$

$F'_{2n} := F'_{2n,0}$

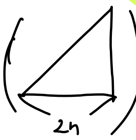


$\mapsto F'_{2n,2k} = U_{2n,2k} - U_{2n,2k+2}$

$k=0$ でつなぐ

$\mapsto f'_{2n,2k} = \frac{1}{2^{2n}} F'_{2n,2k}$

$f'_{2n} := f'_{2n,0}$

$u_{2n} = P\Big($  $\Big)$

$u_{2n,2k} = f'_{2n,2k} + u_{2n,2k+2} = \cdots$

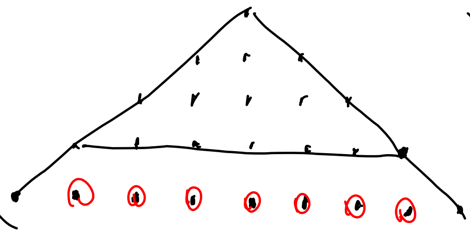
$= f'_{2n,2k} + \cdots + \underbrace{f'_{2n,2n}}_{=1}$

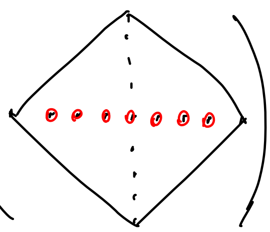
A

$f_0 := 0$

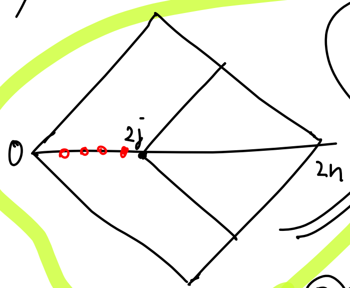
$\boxed{u_{2n} = \sum_{k=0}^{n} f'_{2k}}$

$n=4$

$\frac{1}{2} f_{2n} = P\Big($  $\Big) = \frac{1}{2^{2n}} F'_{2n-2} = \frac{1}{4} f'_{2n-2}$

$f_{2n} := P\Big($  $\Big) = \frac{1}{2} f'_{2n-2}$,

(使ってないかな?)



$f_2 u_{2n-2} + \cdots + f_{2n} u_0 = u_{2n}$

$\boxed{\sum_{i=1}^{k} f_{2i} u_{2k-2i} = u_{2k}}$

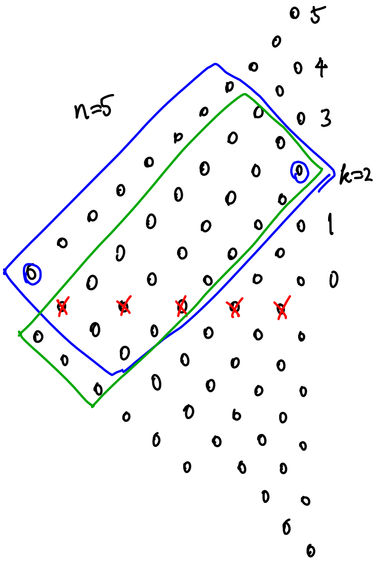
B

induction でつなぐ

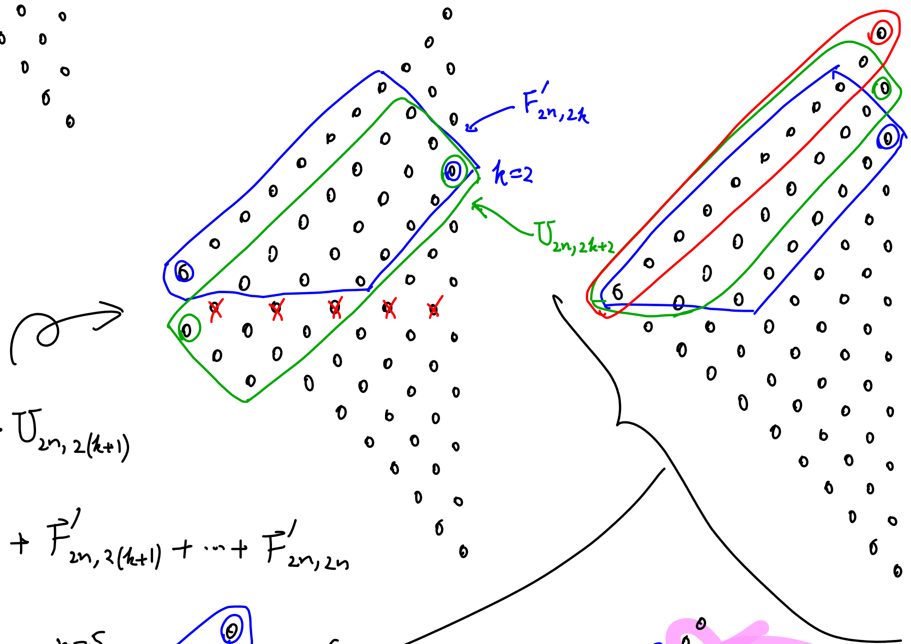
$$F'_{2n,2k} = \binom{2n}{n-k} - \binom{2n}{n-k-1}, \quad U_{2n,2k} = \binom{2n}{n-k}$$

$$= U_{2n,2k} - U_{2n,2k+2}$$

$$U_{2n,2(n+1)} = 0$$



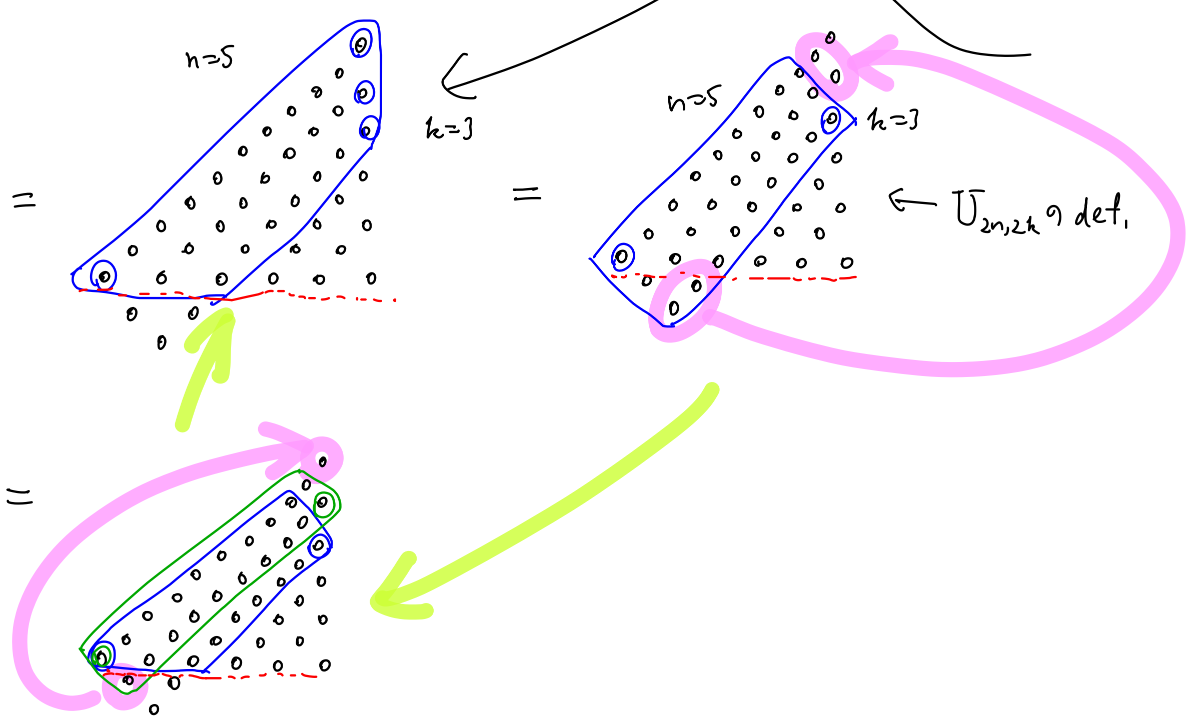
$$U_{2n,2k} = F'_{2n,2k} + U_{2n,2k+2}$$
$$= F'_{2n,2k} + F'_{2n,2k+2} + U_{2n,2k+4}$$
$$= \cdots$$
$$= F'_{2n,2k} + \cdots + F'_{2n,2n} \quad (n-k+1\ 項)$$

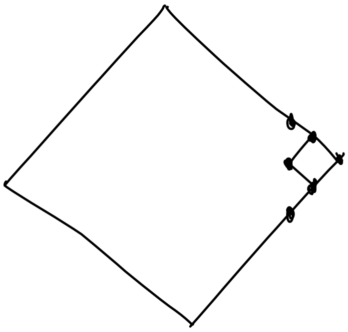


$$U_{2n,2k} = F'_{2n,2k} + U_{2n,2(k+1)}$$

$$U_{2n,2k} = F'_{2n,2k} + F'_{2n,2(k+1)} + \cdots + F'_{2n,2n}$$

$U_{2n,2k} =$





$$u_{2n-2} - u_{2n} = \frac{1}{2^{2n}}\left(4U_{2n-2,0} - U_{2n,0}\right)$$

$$\boxed{\begin{gathered} U_{2n,0} = 2U_{2n-1,2} = 2U_{2n-2,2} + 2U_{2n-2,0} \\ \binom{2n}{n} = 2\binom{2n-1}{n-1} = 2\binom{2n-2}{n-2} + 2\binom{2n-2}{n-1} \end{gathered}}$$

$$= \frac{1}{2^{2n}}\, 2\left(U_{2n-2,0} - U_{2n-2,2}\right)$$

$$= \frac{1}{2^{2n}}\, 2F'_{2n-2,0}$$

$$= \frac{1}{2^{2n}}\, F_{2n}$$

$$= f_{2n}$$

$$\begin{aligned} u_{2n} &= u_{2n-2} - f_{2n} \\ &= u_{2n-4} - f_{2n-2} - f_{2n} \\ &= \cdots \\ &= \underbrace{u_0}_{=1} - f_2 - \cdots - f_{2n} \end{aligned}$$

$$\therefore \boxed{u_{2n} = 1 - \sum_{i=1}^{n} f_{2i}}$$

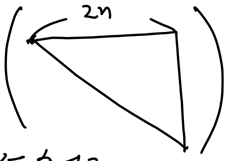
これは証明に使っていない

(注 $u_{2n} = u_{2n-2} - f_{2n}$ には u_{2n}, f_{2n} と u_{2n-2} の"分母"が異なる確率が含まれている　前ページの公式より少し特殊な感じのする公式)

$\eta_{2n} := \#\{ i=0,1,\dots,2n-1 \mid W_i > 0 \text{ or } W_{i+1} > 0 \}$

定理 $P(\eta_{2n}=2k) = u_{2k}u_{2n-2k} \qquad (k=0,1,2,\dots,\ n=k,k+1,\dots)$

証明 kに関する帰納法.

$k=0$のとき $P(\eta_{2n}=0) = P\Bigg($  $\Bigg) \overset{(A)}{=} u_{2n} = u_0 u_{2n}$ で成立. $u_0=1$

(A) ⇐ $\left\{ \begin{array}{l} u_{2n,2k} = f'_{2n,2k} + u_{2n,2n+2} \\ \text{と } u_{2n,2k},\ f'_{2n,2k} \text{ の定義} \end{array} \right.$

$k \geq 1$のとき $k-1$まで成立すると仮定する

$$P(\eta_{2n}=2k) = P(\eta_{2n}=2k, W_1=1) + P(\eta_{2n}=2k, W_1=-1)$$

$$\begin{aligned} P(\eta_{2n}=2k, W_1=1) &= \sum_{j=1}^{k} P(\eta_{2n}=2k, W_1=1, W_2>0, \dots, W_{2j-1}>0, W_{2j}=0) \\ &= \sum_{j=1}^{k} P(W_1=1, W_2>0, \dots, W_{2j-1}>0, W_{2j}=0)\, P(\eta_{2n-2j}=2k-2j) \\ &\overset{\text{by induction}}{=} \sum_{j=1}^{k} \frac{1}{2} f_{2j} \times u_{2k-2j} u_{2n-2k} \overset{(B)}{=} \frac{1}{2} u_{2k}u_{2n-2k} \end{aligned}$$

f_{2j}の定義

(B) ⇐ f_{2j}の定義

$$\begin{aligned} P(\eta_{2n}=2k, W_1=-1) &= \sum_{j=1}^{n-k} P(\eta_{2n}=2k, W_1=-1, W_2<0, \dots, W_{2j-1}<0, W_{2j}=0) \\ &= \sum_{j=1}^{n-k} P(W_1=-1, W_2<0, \dots, W_{2j-1}<0, W_{2j}=0)\, P(\eta_{2n-2j}=2k) \\ &\overset{\text{by induction}}{=} \sum_{j=1}^{n-k} \frac{1}{2} f_{2j} \times u_{2k} u_{2n-2k-2j} \overset{(B)}{=} \frac{1}{2} u_{2k}u_{2n-2k} \end{aligned}$$

f_{2j}の定義

(B) ⇐ f_{2j}の定義

ゆえに, $P(\eta_{2n}=2k) = u_{2k}u_{2n-2k}$

q.e.d.

系 $P(\eta_{2n}=2k) \sim \dfrac{1}{\pi} \dfrac{1}{\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}} \cdot \dfrac{1}{n}$.

証明 Stirlingの公式より, $u_{2k} \sim \dfrac{1}{\cancel{2^{2k}}} \dfrac{\cancel{(2k)^{2k}}\ \cancel{e^{-2k}}\ \sqrt{\cancel{2}\pi \cdot \cancel{2}k}}{\cancel{k^{2k}}\ \cancel{e^{-2k}}\ \cancel{2}\pi k} = \dfrac{1}{\sqrt{\pi k}}$.

同様にして $u_{2n-2k} \sim \dfrac{1}{\sqrt{\pi(n-k)}}$

ゆえに, $P(\eta_{2n}=2k) = u_{2k}u_{2n-2k} \sim \dfrac{1}{\pi} \dfrac{1}{\sqrt{k(n-k)}} = \dfrac{1}{\pi} \dfrac{1}{\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}} \dfrac{1}{n}$. q.e.d.

上の系より, $\displaystyle \lim_{n\to\infty} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P(\eta_{2n}=2k) = \frac{1}{\pi}\int_0^1 f(x) \frac{dx}{\sqrt{x(1-x)}}$ ← 逆正弦法則

$W_n = X_1 + \cdots + X_n$, X_k : i.i.d., $P(X_k = \pm 1) = \frac{1}{2}$

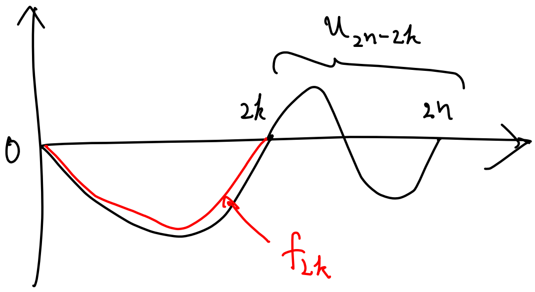
$u_{2n} := P(W_{2n} = 0) = \frac{1}{2^{2n}}\binom{2n}{n}$

$$\binom{-\frac{1}{2}}{n} = \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} = (-1)^n \frac{\overset{2\ 4\ \cdots\ 2n}{1\cdot 3\cdots(2n-1)}}{2^n n!\ \ 2^n n!}$$

$$= (-1)^n \frac{(2n)!}{4^n (n!)^2} = (-1)^n \frac{1}{2^{2n}}\binom{2n}{n} = (-1)^n u_{2n}$$

$$(1-z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n z^{2n} = \sum_{n=0}^{\infty} u_{2n} z^{2n} =: U(z)$$

$f_{2n} := P(W_2 \neq 0, \dots, W_{2n-2} \neq 0, W_{2n} = 0)$



$n = 1, 2, 3, \dots$

$$u_{2n} = f_2 u_{2n-2} + f_4 u_{2n-4} + \cdots + f_{2n} \overbrace{u_0}^{=1}$$

$F(z) := \sum_{n=0}^{\infty} f_{2n} z^{2n}$, $f_0 := 0$.

$$U(z) = \sum_{n=0}^{\infty} u_{2n} z^{2n} = \overset{=1}{u_0} + \underbrace{\sum_{n=1}^{\infty} u_{2n} z^{2n}} = 1 + F(z)U(z)$$

$$F(z)U(z) = \sum_{n=1}^{\infty}\left(\underbrace{\sum_{k=1}^{n} f_{2k} u_{2n-2k}}_{=u_{2n}}\right) z^{2n}$$

$$F(z) = 1 - \frac{1}{U(z)} = 1 - \sqrt{1-z^2} = \cancel{1+}\sum_{\cancel{n=0}\ n=1}^{\infty}\binom{\frac{1}{2}}{n}(-1)^{n-1} z^{2n}$$

$$-\binom{\frac{1}{2}}{n} = \frac{-\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\overbrace{\left(-\frac{1}{2}-n+2\right)}^{-\frac{2n-3}{2}}}{(n!)^2}$$

$$= \frac{1\cdot 3\cdots(2n-3)}{2^n n!}(-1)^n$$

$$f_{2n} = \frac{1\cdot 3\cdots(2n-3)}{2^n n!} = \frac{(2n-2)!}{2^{2n-1} n!(n-1)!}$$

$$= \frac{(2n)!}{2^{2n} n! n!} \times \frac{2n}{2n(2n-1)} = u_{2n} \times \frac{1}{2n-1}$$

$$= u_{2n-2} - u_{2n} \quad (n = 1, 2, \dots)$$

reflection principle でOKか?

$u_{2n-2} - u_{2n}$

$$= \frac{(2n-2)!}{((n-1)!)^2}\frac{1}{4^{n-1}} - \frac{(2n)!}{(n!)^2}\frac{1}{4^n}$$

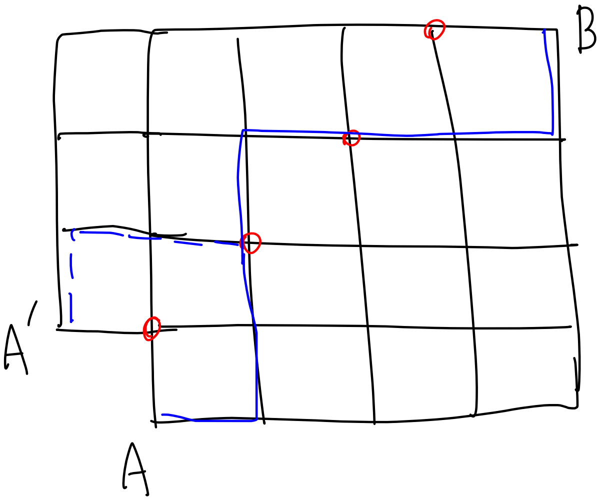
$$= \frac{\overset{2n}{\cancel{4n^2}}}{\cancel{2n}(2n-1)}\frac{(2n)!}{(n!)^2}\frac{1}{4^n} - \frac{(2n)!}{(n!)^2}\frac{1}{4^n}$$

$$= \left(\frac{2n}{2n-1} - 1\right)u_{2n} = \frac{1}{2n-1}u_{2n} = f_{2n}$$

$$\begin{aligned} u_{2n} &= u_{2n-2} - f_{2n} \\ &= u_{2n-4} - f_{2n-2} - f_{2n} \\ &= \cdots \\ &= u_0 - f_2 - \cdots - f_{2n} = 1 - (f_2 + \cdots + f_{2n}) \end{aligned}$$

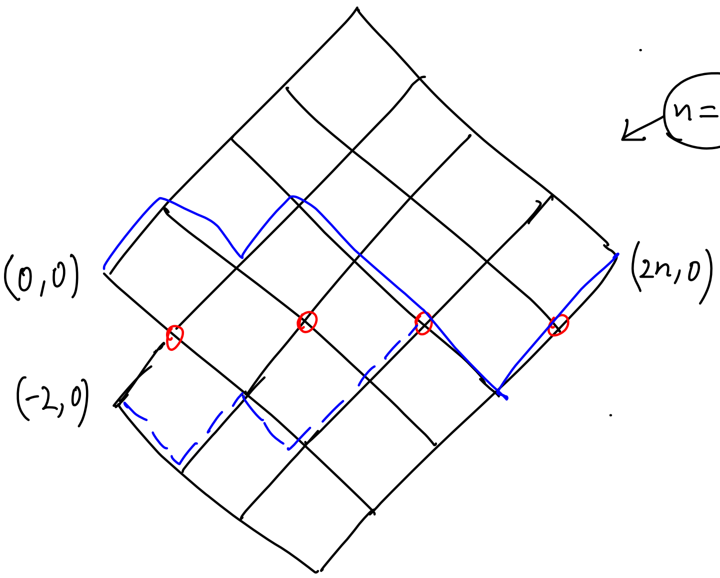
$z^2 U(z) - U(z) = F(z) - u_0 = F(z) - 1$, $\Leftrightarrow F(z) = 1 - (1-z^2)U(z)$

色々OK

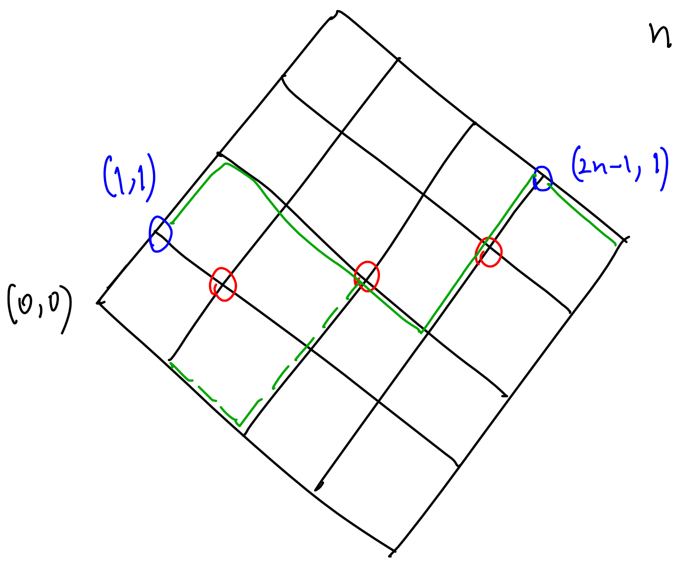


← Catalan 数の話

$$\binom{2n}{n}-\binom{2n}{n-1}=\binom{2n}{n}-\binom{2n}{n}\frac{n}{n+1}$$



n=4



n=4 $\boxed{\text{上半分}}$

$$\binom{2n-2}{n-1}-\binom{2n-2}{n-2}$$

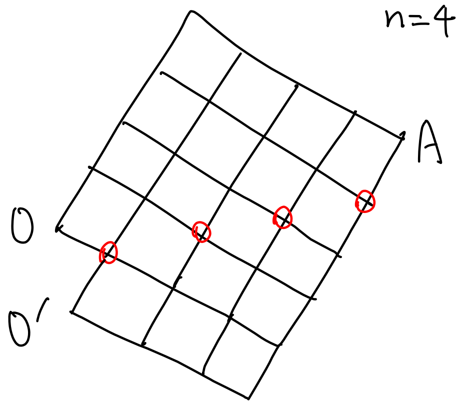
$$=\binom{2n-2}{n-1}-\frac{(2n-2)!}{(n-2)!\,n!}$$

$$=\binom{2n-2}{n-1}-\frac{(2n)!}{(n!)^2}\,\frac{n(n-1)}{2n(2n-1)}$$

$$u_{2n-2}-u_{2n}=\frac{1}{4^{n-1}}\binom{2n-2}{n-1}-\frac{1}{4^n}\binom{2n}{n}=\frac{1}{4^{n-1}}\left(\binom{2n-2}{n-1}-\frac{1}{4}\binom{2n}{n}\right)$$

$$\frac{1}{4}\binom{2n}{n}=\frac{1}{4}\,\frac{(2n)!}{n!\,n!}=\frac{1}{\cancel{4}_{\,2}}\,\frac{(2n-2)!}{(n-2)!\,n!}\times\frac{\cancel{2n}(2n-1)}{\cancel{n}(n-1)}=\frac{(2n-2)!}{(n-2)!\,n!}\,\frac{2n-1}{2(n-1)}=\frac{(2n-1)!}{(n-1)!\,n!}\times\frac{1}{2}$$

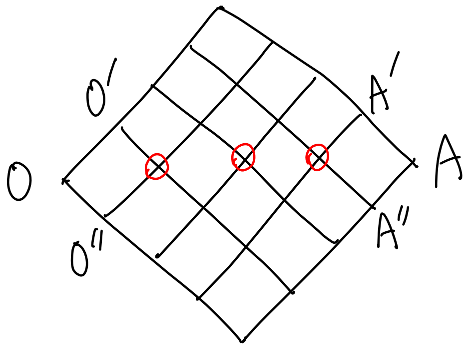
$n=4$



$$\frac{\#(O \to A) - \#(O' - A)}{2^{2n}} =: f'_{2n}$$

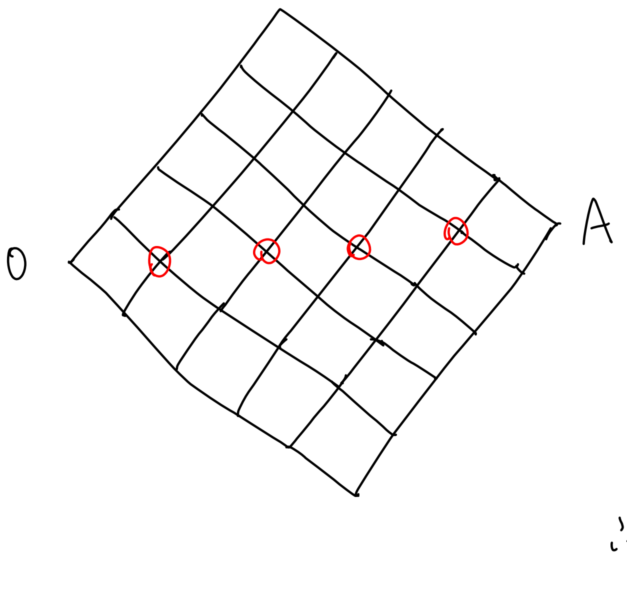
$$f'_{2n} = \frac{1}{2^{2n}}\left(\binom{2n}{n} - \binom{2n}{n-1}\right) = \frac{1}{2^{2n}}\left(\binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n}\right)$$

$$= \frac{1}{n+1}\,\frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{n+1}\,u_{2n}$$



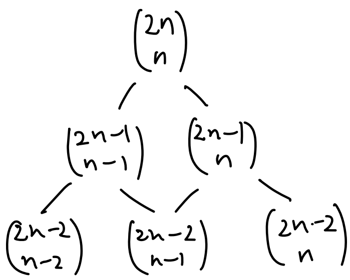
$$f_{2n} = \frac{1}{2^{2n}}\,2 \times 2^{2n-2} f'_{2n-2} = \frac{1}{2} f'_{2n-2} = \frac{1}{2n}\,\frac{1}{2^{2n-2}}\binom{2n-2}{n-1}$$

$$= \frac{1}{2n}\,u_{2n-2}$$



$$u_{2n-2} = \frac{1}{2^{2n-2}}\binom{2n-2}{n-1} = \frac{1}{2^{2n}}\,2^2\binom{2n-2}{n-1}$$

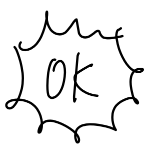
$$u_{2n} = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{2^{2n}}\left(\binom{2n-2}{n-2} + 2\binom{2n-2}{n-1} + \underbrace{\binom{2n-2}{n}}_{=\binom{2n-2}{n-2}}\right)$$



$$\therefore\ u_{2n-2} - u_{2n} = \frac{1}{2^{2n}} \times 2\left(\binom{2n-2}{n-1} - \binom{2n-2}{n-2}\right)$$

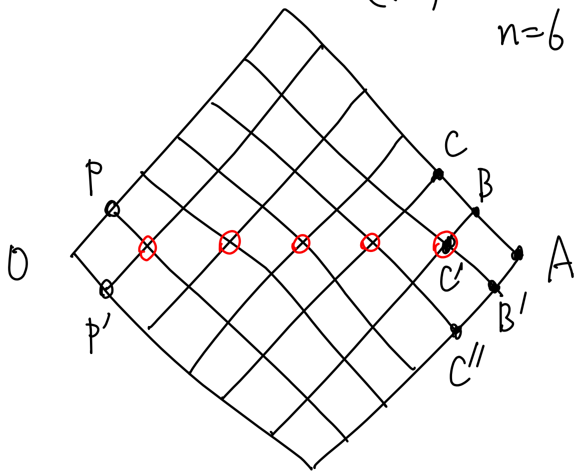
$$= \frac{1}{2}\,\frac{1}{2^{2n-2}}\left(\binom{2n-2}{n-1} - \binom{2n-2}{n-2}\right)$$

$$= \frac{1}{2} f'_{2n-2}$$



$$\binom{2n}{n} = \binom{2n-1}{n-1} + \underbrace{\binom{2n-1}{n}}_{=\binom{2n-1}{n-1}} = 2\binom{2n-1}{n-1} = 2\binom{2n-2}{n-2} + 2\binom{2n-2}{n-1}$$

$n=6$



$$\#(O \to A) = \binom{2n}{n}$$

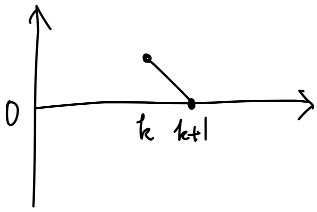
$$\#(O \to A) = \underbrace{\#(O \to B)}_{\substack{\shortparallel \\ \binom{2n-1}{n-1}}} + \underbrace{\#(O \to B')}_{\substack{\shortparallel \\ \binom{2n-1}{n}}}$$

$$\#(O \to B) = \underbrace{\#(O \to C)}_{\substack{\shortparallel \\ \binom{2n-2}{n-2}}} + \underbrace{\#(O \to C')}_{\substack{\shortparallel \\ \binom{2n-2}{n-1}}}$$

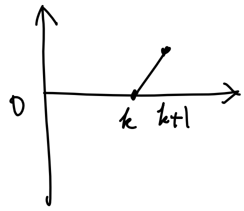
$$\#(O \to B') = \underbrace{\#(O \to C')}_{=\binom{2n-2}{n-1}} + \underbrace{\#(O \to C'')}_{=\binom{2n-2}{n}}$$

Ok

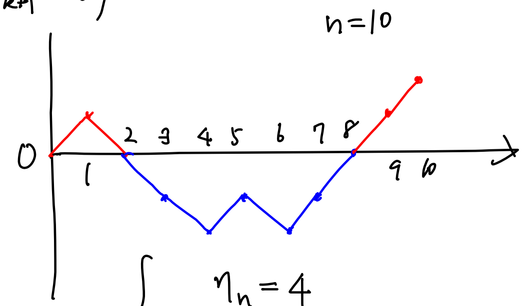
$\eta_n :=$ (ういている時間) $= \#\{k=0,1,\dots,n-1 \mid W_k>0 \text{ or } W_{k+1}>0\}$



と



もOK,

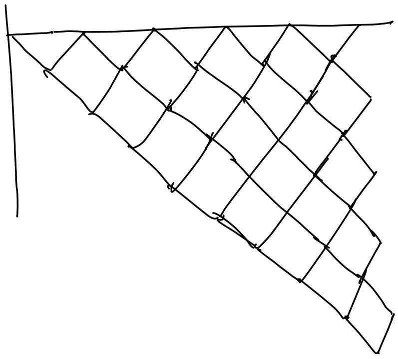


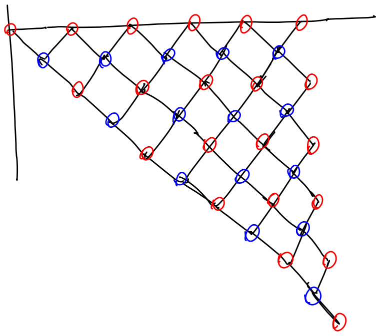
$\begin{cases} \eta_n = 4 \\ 10-\eta_n = 6 \end{cases}$

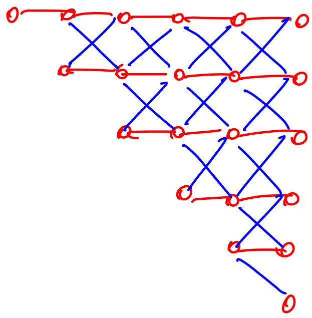
$P(\eta_{2n}=2k) = u_{2k}u_{2n-2k}$ を示したい

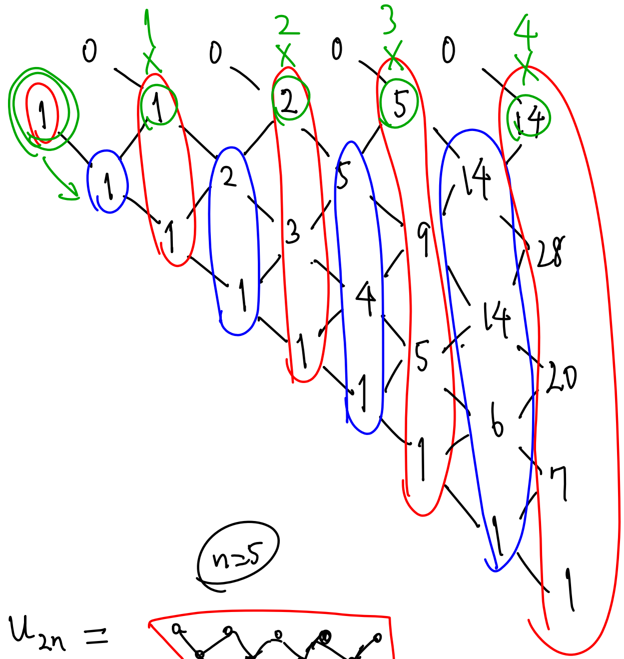
対称性より, $P(\eta_{2n}=2n-2k) = P(\eta_{2n}=2k)$

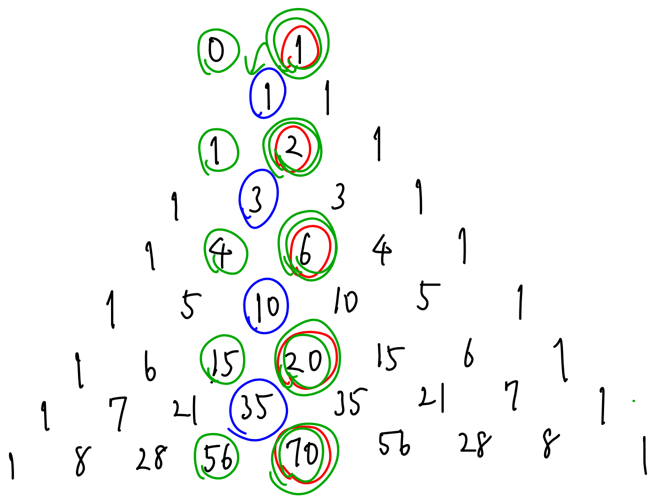
$\boxed{k=0\text{のとき}}$



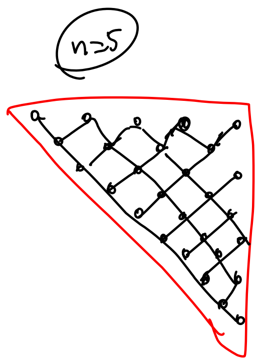








$u_{2n} =$



$\binom{2n}{n} = 2\binom{2n-1}{n-1}$, $\binom{2n-1}{n-1} =$

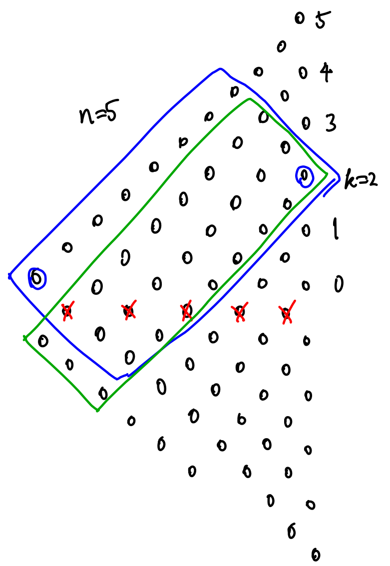
$\boxed{k=0\text{ でOK}}$

これは $u_{2n} = f'_{2n,0} + f'_{2n,2} + \cdots + f'_{2n,2n}$ より

$$F'_{2n,2k} = \binom{2n}{n-k} - \binom{2n}{n-k-1}, \qquad U_{2n,2k} = \binom{2n}{n-k}$$

$$= U_{2n,2k} - U_{2n,2k+2}$$

$$U_{2n,2(n+1)} = 0$$

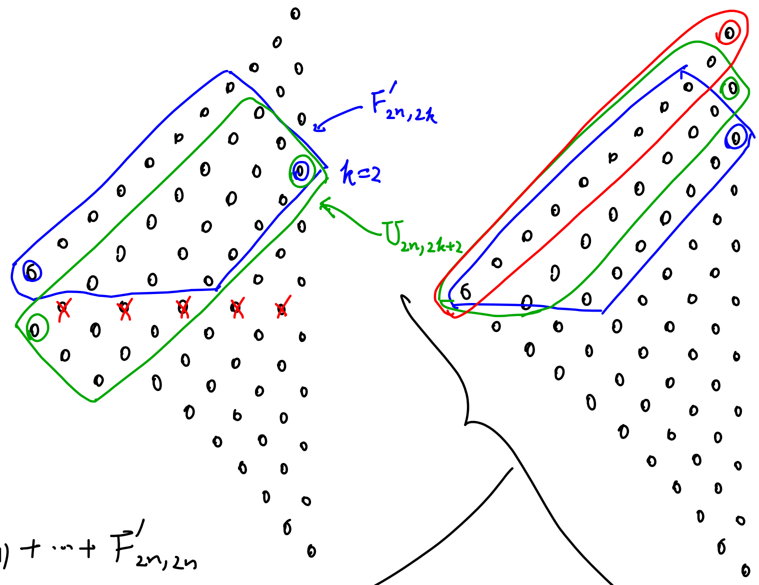


$$U_{2n,2k} = F'_{2n,2k} + U_{2n,2k+2}$$

$$= F'_{2n,2k} + F'_{2n,2k+2} + U_{2n,2k+4}$$

$$= \cdots$$

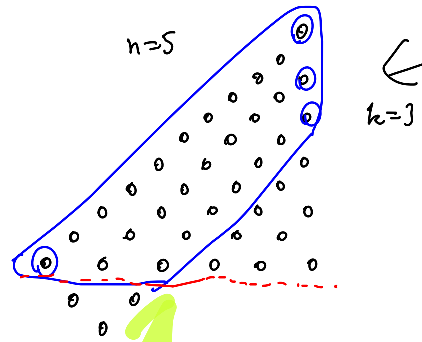
$$= F'_{2n,2k} + \cdots + F'_{2n,2n} \quad (n-k+1 \text{ 項})$$



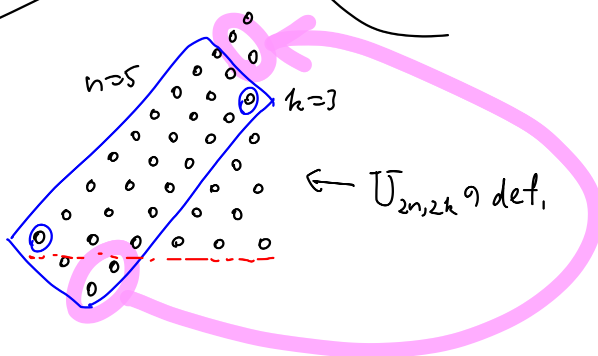
$$U_{2n,2k} = F'_{2n,2k} + U_{2n,2(k+1)}$$

$$U_{2n,2k} = F'_{2n,2k} + F'_{2n,2(k+1)} + \cdots + F'_{2n,2n}$$

$U_{2n,2k} =$

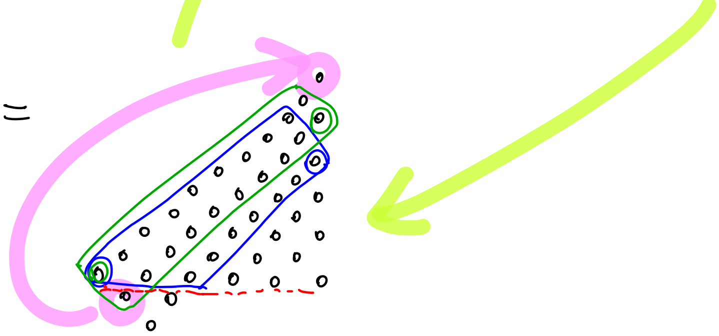


$=$



← $U_{2n,2k}$ の def.

$=$



$\boxed{k-1 \text{までOKと仮定する}}$

$$P(\eta_{2n}=2k) = P(\eta_{2n}=2k, W_1=1) + P(\eta_{2n}=2k, W_1=-1)$$

$$\begin{aligned} P(\eta_{2n}=2k, W_1=1) = {} & P(\eta_{2n}=2k, W_1=1, W_2=0) \\ & + P(\eta_{2n}=2k, W_1=1, W_2>0, W_3>0, W_4=0) \\ & + \cdots \\ & + P(\eta_{2n}=2k, W_1=1, W_2>0, \dots, W_{2k-1}>0, W_{2k}=0) \end{aligned}$$

$$\begin{aligned} P(\eta_{2n}=2k, W_1=-1) = {} & P(\eta_{2n}=2k, W_1=-1, W_2=0) \\ & + P(\eta_{2n}=2k, W_1=-1, W_2<0, W_3<0, W_4=0) \\ & + \cdots \\ & + P(\eta_{2n}=2k, W_1=-1, W_2<0, \dots, W_{2(n-k)-1}<0, W_{2(n-k)}=0) \end{aligned}$$

$$P(\eta_{2n}=2k, W_1=1, W_2>0, \dots, W_{2j-1}>0, W_{2j}=0) \quad \leftarrow j=1,2,\dots,k$$

$$\begin{aligned} &= P(W_1=1, W_2>0, \dots, W_{2j-1}>0, W_{2j}=0) \times P(\eta_{2n-2j}=2k-2j) \\ &= \frac{1}{2} f_{2j} u_{2k-2j} u_{2n-2k} \quad (\text{by induction}) \end{aligned}$$

$$P(\eta_{2n}=2k, W_1=-1, W_2<0, \dots, W_{2j-1}<0, W_{2j}=0) \quad \leftarrow j=1,2,\dots,n-k,$$

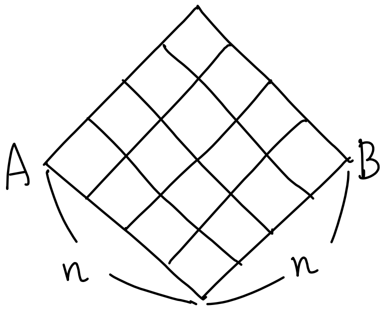
$$\begin{aligned} &= P(W_1=-1, W_2<0, \dots, W_{2j-1}<0, W_{2j}=0)\, P(\eta_{2n-2j}=2k) \\ &= \frac{1}{2} f_{2j} u_{2k} u_{2n-2k-2j} \quad (\text{by induction}) \end{aligned}$$

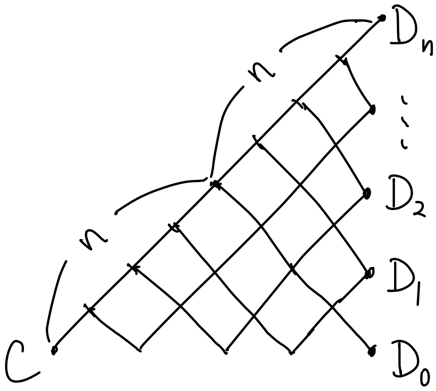
$$\sum_{j=1}^{k} f_{2j} u_{2k-2j} = u_{2k}, \quad \sum_{j=1}^{n-k} f_{2j} u_{2n-2k-2j} = u_{2n-2k} \text{ より}$$

$$P(\eta_{2n}=2k) = \frac{1}{2} u_{2k} u_{2n-2k} + \frac{1}{2} u_{2k} u_{2n-2k} = u_{2k} u_{2n-2k}.$$

$\boxed{\text{q.e.d.}}$

逆正弦法則に向けて (1)





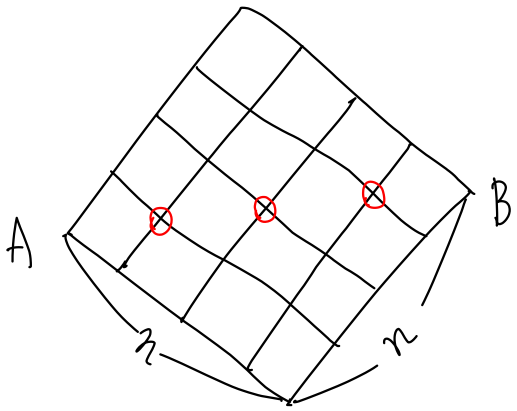
(この図では $n=4$)

0以上の整数nについて

$$(\text{AからBへの最短経路の数}) = \sum_{j=0}^{n} (\text{CからD}_j\text{への最短経路の数})$$

となる(ことを示せ).

逆正弦法則に向けて (2)



$$F_{2n} = \left(\begin{array}{l}\text{AからBへの○を通らない}\\ \text{最短経路の数}\end{array}\right)$$

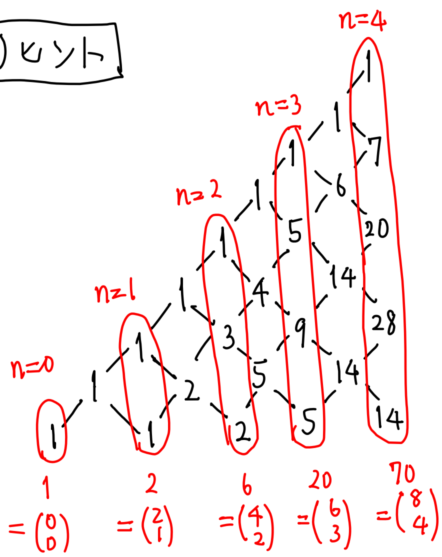
$$U_{2n} = (\text{AからBへの最短経路の数})$$

とおく. このとき

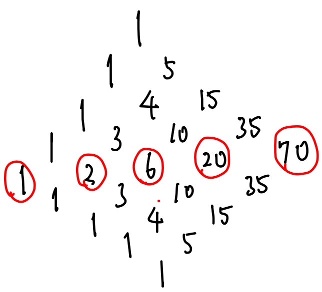
$$U_{2n} = \sum_{j=1}^{n} F_{2j}\, U_{2n-2j}$$

となる(ことを示せ).

(1) ヒント

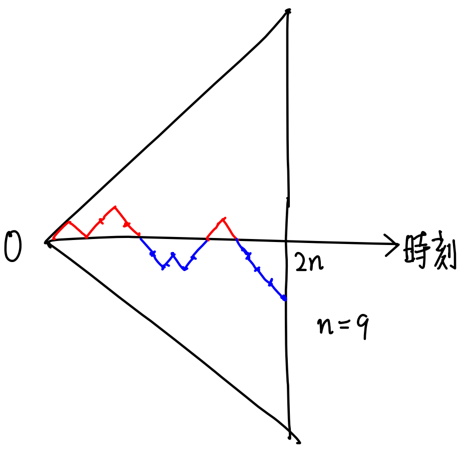


Pascalの三角形の一部



逆正弦法則

右ななめ45°方向に1ステップずつ進む。



左図では時刻が1〜2nのあいだで浮いている時間 η_{2n} は 8 になる。

$\eta_{2n}=2k$ となる時刻 2n までの経路の数を $N(\eta_{2n}=2k)$ と書き、$U_{2k}=\binom{2k}{k}=\frac{(2k)!}{(k!)^2}$ とおく。

逆正弦法則 I　$N(\eta_{2n}=2k)=U_{2k}U_{2n-2k}$.

逆正弦法則 II　$n\to\infty$ のとき、

$$2^{-2n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)N(\eta_{2n}=2k)\longrightarrow\frac{1}{\pi}\int_0^1\frac{f(x)\,dx}{\sqrt{x(1-x)}},$$

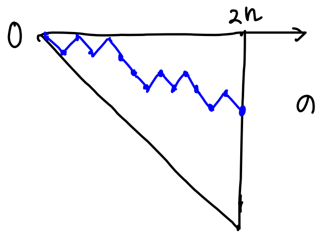
$\left(\begin{array}{c}\text{浮いている時間の割合}\\ \eta_{2n}/(2n)\text{ の分布}\end{array}\right)\longrightarrow$ (逆正弦分布).

I ⇒ II　Wallisの公式 $\frac{1}{2^{2k}}U_{2k}\sim\frac{1}{\sqrt{\pi k}}$ と I より、$2^{-2n}N(\eta_{2n}=2k)\sim\frac{1}{\pi\sqrt{k(n-k)}}$. ゆえに

$$2^{-2n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)N(\eta_{2n}=2k)\sim\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\frac{1}{\pi\sqrt{k(n-k)}}=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\frac{1/n}{\pi\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}}\to\int_0^1 f(x)\frac{dx}{\pi\sqrt{x(1-x)}}.\quad\square$$

I の証明　k に関する帰納法。

k=0 の場合　$N(\eta_{2n}=0)=U_{2n}$ を示せばよい。$\eta_{2n}=0$ (浮いている時間がない)の経路は

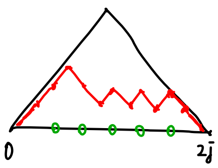


のような経路。ゆえに (1) より $N(\eta_{2n}=0)=U_{2n}$.

$0\leqq j<k$ のとき $N(\eta_{2n}=2j)=U_{2j}U_{2n-2j}$ と仮定した場合

$\eta_{2n}=2k$ でかつ最初に右上に進む経路の数を求めよう。γ はそのような経路とする。γ が最初に右上に進んだ後に時間軸に初めて触れる時刻を $2j$ とする。γ の経路で浮いている時間の長さは $2k$ なので $1\leqq j\leqq k$ でなければいけない。γ を時刻 $2j$ 以下に $2j$ 以上に分割する。

時刻 $2j$ 以下の部分は



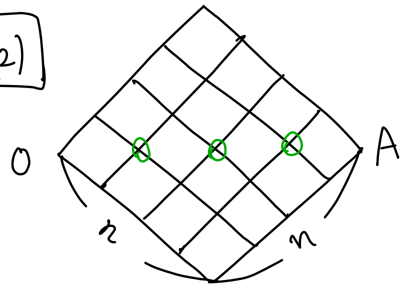
の形で ○ を通らない経路になる。(2) における定義より、そのような経路の数は $\frac{1}{2}F_{2j}$ になる。時刻 $2j$ 以上の部分の形は $N(\eta_{2n-2j}=2k-2j)$ 通りある。帰納法の仮定より、それは $U_{2k-2j}U_{2n-2k}$ に等しい。ゆえに γ の個数は (2) より

$\frac{1}{2}\sum_{j=1}^{k}F_{2j}U_{2k-2j}U_{2n-2k}=\frac{1}{2}U_{2k}U_{2n-2k}$ となる。最初に右下に進む経路の数も同様にして

$\frac{1}{2}U_{2k}U_{2n-2k}$ となる。したがって、$N(\eta_{2n}=2k)$ はそれらの和で求める結果を得る。

q.e.d.

$\boxed{(2)}$



$$\begin{cases} U_{2n} := (\text{O から A への最短経路の数}), \\ F_{2n} := (\text{O から A への ○ を通らない最短経路の数}). \end{cases}$$

O から A には $2n$ ステップで到着する。
そのあいだに右上と右下に進むステップがどちらも n 回になる。

このことから、$U_{2n} = (2n\text{個から}n\text{個選ぶ組み合わせの数}) = \binom{2n}{n} = \dfrac{(2n)!}{(n!)^2}$.

このとき、$\displaystyle U_{2n} = \sum_{j=1}^{n} F_{2j} U_{2n-2j}$. ← (2)　　(注) $U_0 = \binom{0}{0} = 1$.

$\boxed{証明}$ 任意に O から A への最短経路 γ をとる。γ が O 以外の直線 OA 上の点(上の図の ○ と点 A)にもどってくる最初の時刻を $2j$ とする。$1 \leqq j \leqq n$ となる。

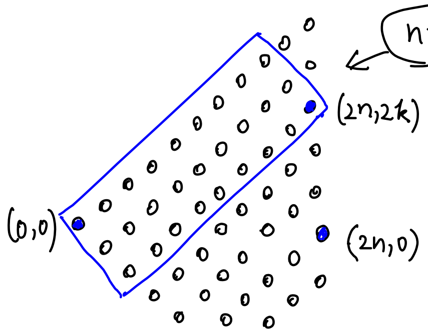
γ を時刻が $2j$ 以下と $2j$ 以上の部分に分割する。

$2j$ 以下の部分の形の場合の数は F_{2j} である。

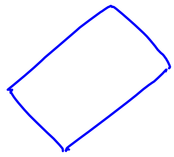
$2j$ 以上の部分の形の場合の数は U_{2n-2j} である

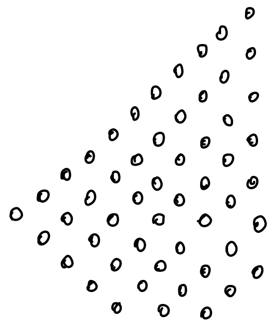
したがって、$\displaystyle U_{2n} = \sum_{j=1}^{n} F_{2j} U_{2n-2j}$ となる　　$\boxed{q.e.d.}$

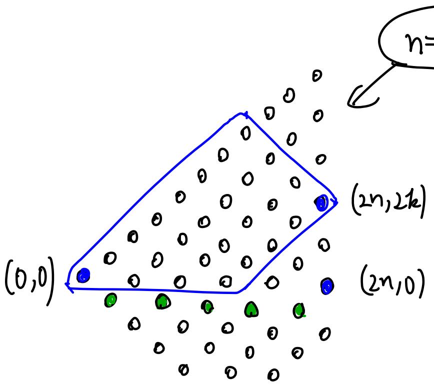
$\boxed{(1)}$ 左端から右上または右下にすすむ.



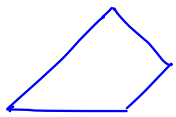
$U_{2n,2k} := \left(\begin{array}{l}(0,0)\text{から}(2n,2k)\text{への}\\ \text{経路の数}\end{array}\right) = \binom{2n}{n-k}$.

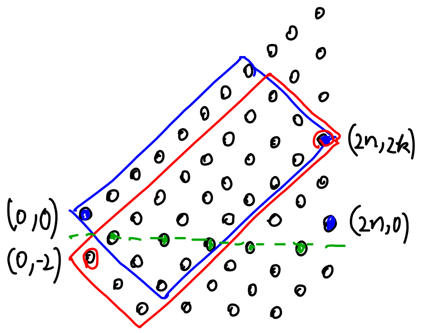
=  と書く.

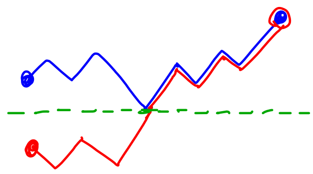




$G_{2n,2k} := \left(\begin{array}{l}(0,0)\text{から}(2n,2k)\text{への経路で}\\ \text{● を通らないものの数}\end{array}\right)$

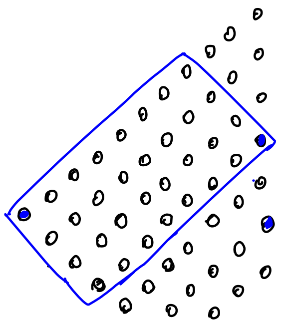
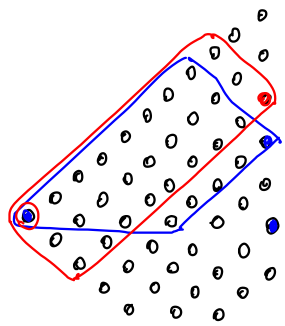
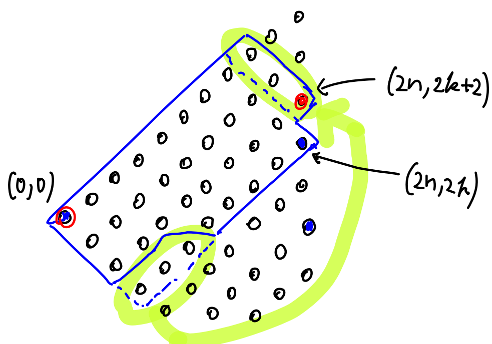
=  と書く.



反射原理  より,

(0,0) から (2n,2k) への経路で ● のどれかを通るものと
(0,-2) から (2n,2k) への経路が一対一に対応する. ゆえに,

$$G_{2n,2k} = U_{2n,2k} - U_{2n,2k+2}.$$

 $\cong$  = 

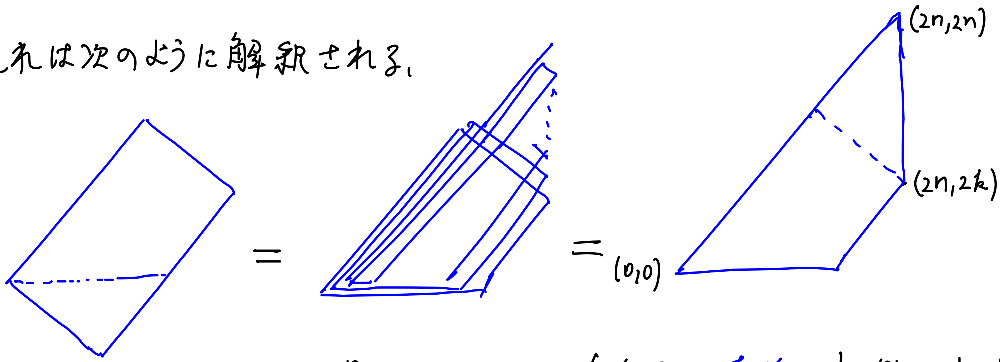
$U_{2n,2k} = G_{2n,2k} + U_{2n,2k+2} = \left(\begin{array}{l}\text{右図の青線の内側を通って}(0,0)\text{から}\\ (2n,2k)\text{または}(2n,2k+2)\text{に行く経路の数}\end{array}\right)$

$U_{2n,2k} = G_{2n,2k} + U_{2n,2k+2} = G_{2n,2k} + G_{2n,2k+2} + U_{2n,2k+4} = \cdots$

$= G_{2n,2k} + G_{2n,2k+2} + \cdots + G_{2n,2n}$

（注 $U_{2n,2n+2} = 0$.）

これは次のように解釈される。



$U_{2n,2k} = \sum_{j=k}^{n} G_{2n,2j} =$ （右図で青線の内側を点(0,0)から点(2n,2k),…,(2n,2n)のどれかに行く経路の数）

特に $U_{2n} = U_{2n,0} = \sum_{j=0}^{n} G_{2n,2j} =$ （下図で青線の内側を左端(0,0)から右端の点のどれかに行く経路の数） (1)