一般化されたLaplaceの方法

$l_i>0$, $a_i>0$, $b_i>0$ とし, $\tilde{Z}_n$ を次のように定める:

$$\tilde{Z}_n := \int_0^{l_1} dx_1 \cdots \int_0^{l_d} dx_d \exp\left(-n x_1^{a_1} \cdots x_d^{a_d}\right) x_1^{b_1-1} \cdots x_d^{b_d-1}.$$

$x_i = X_i^{1/a_i}$ とおくと,

$$x_i^{b_i-1} dx_i = x_i^{b_i} \frac{dx_i}{x_i} = X_i^{\frac{b_i}{a_i}} \cdot \frac{1}{a_i} \frac{dX_i}{X_i} = \frac{1}{a_i} X_i^{\lambda_i - 1} dX_i, \quad \lambda_i := \frac{b_i}{a_i}$$

なので, $L_i = l_i^{a_i}$ とおくと,

$$\tilde{Z}_n = \frac{1}{a_1 \cdots a_d} Z_n, \quad Z_n = \int_0^{L_1} dx_1 \cdots \int_0^{L_d} dx_d \exp\left(-n x_1 \cdots x_d\right) x_1^{\lambda_1 - 1} \cdots x_d^{\lambda_d - 1}.$$

以下を仮定する:

$$\lambda := \lambda_1 = \cdots = \lambda_m < \lambda_{m+1} \leqq \cdots \leqq \lambda_d, \quad \lambda = \min\{\lambda_1, \ldots, \lambda_d\}, \ m \text{ は } \lambda \text{ の重複度}.$$

次を示したい:

$$-\log Z_n = \lambda \log n - (m-1) \log\log n + O(1).$$

$x_1 = \dfrac{t}{n x_2 \cdots x_d}$ とおき，積分変数を $(x_1, x_2, \ldots, x_d)$ から $(t, x_2, \ldots, x_d)$ に変換したい．

$(x_1, \ldots, x_d)$ は次の範囲を動く：$0 < x_i < L_i$ $(i=1,\ldots,d)$． これは次と同値：

$$0 < \frac{t}{n x_2 \cdots x_d} < L_1, \quad 0 < x_2 < L_2, \ \ldots, \ 0 < x_d < L_d.$$

これはさらに次と同値：

$$0 < x_{m+1} < L_{m+1}, \ \ldots, \ 0 < x_d < L_d,$$

$$0 < t < n L_1 \cdots L_m x_{m+1} \cdots x_d, \quad \longleftarrow \quad \frac{t}{n L_1 \cdots L_m} < x_{m+1} \cdots x_d$$

$$\frac{t}{n L_1 L_3 \cdots L_m x_{m+1} \cdots x_d} < x_2 < L_2, \quad \longleftarrow \quad \frac{t}{n L_1 L_3 \cdots L_m} < x_{m+1} \cdots x_d x_2$$

$$\frac{t}{n L_1 x_2 L_4 \cdots L_m x_{m+1} \cdots x_d} < x_3 < L_3, \quad \longleftarrow \quad \frac{t}{n L_1 L_4 \cdots L_m} < x_{m+1} \cdots x_d x_2 x_3$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{t}{n L_1 x_2 \cdots x_{m-1} x_{m+1} \cdots x_d} < x_m < L_m \quad \longleftarrow \quad \frac{t}{n L_1} < x_{m+1} \cdots x_d x_2 \cdots x_{m-1}$$

$$x_1^{\lambda_1 - 1} \cdots x_d^{\lambda_d - 1} dx_1 \cdots dx_d = \left(\frac{t}{n x_2 \cdots x_d}\right)^{\lambda_1 - 1} x_2^{\lambda_2 - 1} \cdots x_d^{\lambda_d - 1} \frac{1}{n x_2 \cdots x_d} dt\, dx_2 \cdots dx_d$$

$$= n^{-\lambda_1} t^{\lambda_1 - 1} x_2^{\lambda_2 - \lambda_1 - 1} \cdots x_d^{\lambda_d - \lambda_1 - 1} dt\, dx_2 \cdots dx_d$$

$$= n^{-\lambda} t^{\lambda - 1} x_2^{-1} \cdots x_m^{-1} x_{m+1}^{\lambda_{m+1} - \lambda - 1} \cdots x_d^{\lambda_d - \lambda - 1} dt\, dx_2 \cdots dx_d$$

$$Z_n = \int_0^{L_{m+1}} dx_{m+1} \cdots \int_0^{L_d} dx_d \int_0^{n L_1 \cdots L_d x_{m+1} \cdots x_d} dt\ e^{-t}$$

$$\times \int_{\frac{t}{n L_1 L_3 \cdots L_d x_{m+1} \cdots x_d}}^{L_2} dx_2 \int_{\frac{t}{n L_1 L_4 \cdots L_d x_{m+1} \cdots x_d x_2}}^{L_3} dx_3 \cdots \int_{\frac{t}{n L_1 x_{m+1} \cdots x_d x_2 \cdots x_{m-1}}}^{L_m} dx_m$$

$$\times n^{-\lambda_1} t^{\lambda_1 - 1} x_2^{-1} \cdots x_m^{-1} x_{m+1}^{\lambda_{m+1} - \lambda - 1} \cdots x_d^{\lambda_d - \lambda - 1}$$

$$\int_{\frac{t}{nL_1L_3\cdots L_m x_{m+1}\cdots x_d}}^{L_2} dx_2 \int_{\frac{t}{nL_1L_4\cdots L_m x_{m+1}\cdots x_d x_2}}^{L_3} dx_3 \cdots \int_{\frac{t}{nL_1 x_{m+1}\cdots x_d x_2\cdots x_{m-1}}}^{L_m} dx_m \; x_2^{-1}\cdots x_m^{-1}$$

$$= x_2^{-1}\cdots x_{m-1}^{-1}$$

$$\times\left(\log\left(nL_1L_m x_{m+1}\cdots x_d x_2\cdots x_{m-1}\right) - \log t\right)$$

$$= x_2^{-1}\cdots x_{m-1}^{-1}\log n\,(1+o(1))$$

$$= x_2^{-1}(\log n)^{m-3}\left(\log\left(nL_1L_3\cdots L_m x_{m+1}\cdots x_d x_2\cdots x_{m-1}\right) - \log t\right)(1+o(1))$$

$$= x_2^{-1}(\log n)^{m-2}(1+o(1))$$

$$= (\log n)^{m-2}\left(\log\left(nL_1\cdots L_m x_{m+1}\cdots x_d\right) - \log t\right)(1+o(1))$$

$$= (\log n)^{m-1}(1+o(1))$$

$$\int_0^{nL_1\cdots L_d x_{m+1}\cdots x_d} dt\; e^{-t}\,t^{\lambda-1} = \Gamma(\lambda)\,(1+o(1)).$$

Const. >0

$$\int_0^{L_{m+1}} dx_{m+1}\cdots\int_0^{L_d} dx_d\; x_{m+1}^{\lambda_{m+1}-\lambda-1}\cdots x_d^{\lambda_d-\lambda-1} = \left[\frac{x_{m+1}^{\lambda_{m+1}-\lambda}}{\lambda_{m+1}-\lambda}\right]_0^{L_{m+1}}\cdots\left[\frac{x_d^{\lambda_d-\lambda}}{\lambda_d-\lambda}\right]_0^{L_d} = \underbrace{\frac{L_{m+1}^{\lambda_{m+1}-\lambda}}{\lambda_{m+1}-\lambda}\cdots\frac{L_d^{\lambda_d-\lambda}}{\lambda_d-\lambda}}$$

以上をまとめると,

$$Z_n = \mathrm{Const.}\; n^{-\lambda}(\log n)^{m-1}(1+o(1))$$

$$\mathrm{Const.} = \Gamma(\lambda)\prod_{i=m+1}^{d}\frac{L_i^{\lambda_i-\lambda}}{\lambda_i-\lambda}$$

すなわち,

$$-\log Z_n = \lambda\log n - (m-1)\log\log n + \mathrm{const.} + o(1).$$